

A NEW DEFORMED SUPERSYMMETRIC OSCILLATOR

S.Meljanac¹, M.Mileković^{2,+} and A.Perica^{1,++}

¹ *Rudjer Bošković Institute, Bijenička c.54, 41001 Zagreb, Croatia*

² *Prirodoslovno-Matematički Fakultet, Zavod za teorijsku fiziku,
Bijenička c.32, 41000 Zagreb, Croatia*

⁺ *e-mail: marijan@phy.hr*

⁺⁺ *e-mail: perica@thphys.irb.hr*

Abstract

We construct and discuss the Fock-space representation for a deformed oscillator with "peculiar" statistics. We show that corresponding algebra represents deformed supersymmetric oscillator.

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Introduction. - The subject of quantum statistics, which is different from the ordinary Bose and Fermi statistics, has attracted much attention in the past few years. One motivation comes from the study of some phenomena in condensed matter where the dynamics is essentially two-dimensional, thus allowing anyon-like statistics [1]. The other motivation comes from the theoretical and experimental search for possible violation of the Pauli exclusion principle in four dimensions [2], where quon-like statistics [3] might play a significant role.

In either case, quantum groups and algebras [4] have offered a new insight into the subject. The introduction of q -deformations of the Heisenberg-Weyl algebras has led to the investigation of particles interpolating between bosons and fermions, [5]. The q -bosons have been introduced and discussed in a variety of ways [6]. Particularly useful formulations of associative q -boson algebras are proposed through the Yang-Baxter R -matrix [7] which generalizes the notion of permutational symmetry. The simplest such algebras, associated to 4×4 R -matrices, were investigated to some extent in [8] and three types of deformed algebras were found, among them a "peculiar" algebra which corresponded to the R -matrix of the eight-vertex form. The detailed structure of the "peculiar" algebra was not discussed in [8] and it remains unclear whether this algebra makes a sense physically, i.e. whether there exists a well-defined Fock-space representation with positive norms and number operators.

In this Letter we investigate the structure of this "peculiar" algebra. We construct and discuss the corresponding Fock-space representation and show that norms of all states are positive definite. We show that this algebra represents a new kind of deformed supersymmetric oscillator.

"Peculiar" algebra. - We start with the following ("peculiar") oscillator algebra [8]:

$$\begin{aligned}
(1 - \epsilon q) a_1^2 &= \epsilon' (1 + \epsilon q) a_2^2, \\
a_1 a_2 &= \epsilon'' a_2 a_1, \\
a_1 a_1^\dagger &= 1 + (\epsilon q^{-1} + \frac{q^{-2} - 1}{2}) a_1^\dagger a_1 + (\frac{q^{-2} - 1}{2}) a_2^\dagger a_2, \\
a_2 a_2^\dagger &= 1 + (-\epsilon q^{-1} + \frac{q^{-2} - 1}{2}) a_2^\dagger a_2 + (\frac{q^{-2} - 1}{2}) a_1^\dagger a_1, \\
a_1 a_2^\dagger &= (\epsilon'' \frac{1 + q^{-2}}{2}) a_2^\dagger a_1 + (\epsilon' \frac{q^{-2} - 1}{2}) a_1^\dagger a_2, \\
a_2 a_1^\dagger &= (\epsilon'' \frac{1 + q^{-2}}{2}) a_1^\dagger a_2 + (\epsilon' \frac{q^{-2} - 1}{2}) a_2^\dagger a_1,
\end{aligned} \tag{1}$$

where $q \in \mathbf{R}$, $\epsilon^2 = \epsilon'^2 = \epsilon''^2 = 1$. When $q^2 = 1$, the above algebra, eq.(1), represents one Bose and one Fermi oscillator which commute or anticommute (depending on whether ϵ'' is 1 or -1). We observe that the "peculiar" algebra, eq.(1), has no well-defined number operators N_1, N_2 , in the usual sense: $[N_i, a_j] = -a_i \delta_{ij}$, $[N_i, a_j^\dagger] = a_i \delta_{ij}$, $i, j = 1, 2$. From $[N_1, a_1] = -a_1$, it follows that $[N_1, a_1^2] = -2a_1^2$. Owing to eq.(1) one obtains $[N_1, a_2^2] = -2a_2^2$, which contradicts the demanded relation $[N_1, a_2] = 0$. Hence the relations $[N_1, a_2] = [N_2, a_1] = 0$ contradict eq.(1). However, the total number operator $N = N_1 + N_2$ is well defined. Of course, when $q^2 = 1$, the number operators N_1 and N_2 are also well defined, i.e. $N_{1,2} = N_{B,F}$.

Fock-space representation of "peculiar" algebra. - Let us assume that there is a vacuum $|0, 0\rangle$ satisfying $a_i |0, 0\rangle = 0$, $i = 1, 2$. The excited states can be

constructed by multiple action of the operators a_1^\dagger and a_2^\dagger on the vacuum $|0, 0\rangle$ and are of the form

$$|n_1, n_2\rangle \propto (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} |0, 0\rangle, \quad n_1, n_2 \in \mathbf{N}. \quad (2)$$

Notice that the action of N_1 (N_2) on the states (2) is not well defined for $n_2 \geq 2$ ($n_1 \geq 2$) and hence, in general, n_1 (n_2) are not eigenvalues of N_1 (N_2). This is the consequence of the quadratic relation $a_1^2 \propto a_2^2$ (eq.(1)) for $q^2 \neq 1$. Furthermore, we find that the $|n_1, n_2\rangle$ states are degenerate (linearly dependent) for the fixed sum $n_1 + n_2 = n$, in the following sense: $|n_1, n_2\rangle \propto |n_1 - 2k, n_2 + 2k\rangle$, $k \in \mathbf{Z}$, $n_1 - 2k \geq 0, n_2 + 2k \geq 0$ and $|n_1 + 1, n_2 - 1\rangle \propto |n_1 + 1 - 2k, n_2 - 1 + 2k\rangle$, $k \in \mathbf{Z}, n_1 + 1 - 2k \geq 0, n_2 - 1 + 2k \geq 0$. The states for fixed n can be reduced to two states, $|n, 0\rangle$ and $|(n-1), 1\rangle$ or alternatively, to $|0, n\rangle$ and $|1, (n-1)\rangle$. Hence, the complete set of states can be represented by two symmetric pictures (for $q^2 \neq 1$)

$$|n, \nu\rangle \propto (a_1^\dagger)^n (a_2^\dagger)^\nu |0, 0\rangle, \quad (a) \quad (3)$$

$$|\nu, n\rangle \propto (a_1^\dagger)^\nu (a_2^\dagger)^n |0, 0\rangle, \quad (b)$$

where $n \in \mathbf{N}_0$, $\nu = 0, 1$. Now, n (ν) is eigenvalue of N_1 (N_2) in the picture (3a) i.e. N_2 (N_1) in the picture (3b). In the following we use the first picture (3a).

There are two towers of states generated by the a_1^\dagger creation operator. One tower is $|n, 0\rangle$, generated from the $|0, 0\rangle$ vacuum, ($\nu = 0$). The other tower is $|n, 1\rangle$, generated from the second vacuum $|0, 1\rangle$, ($\nu = 1$). Using the algebra (1) we find that

$$a_1^\dagger a_1 (a_1^\dagger)^n (a_2^\dagger)^\nu |0, 0\rangle = \phi_1(n, \nu) (a_1^\dagger)^n (a_2^\dagger)^\nu |0, 0\rangle,$$

$$\phi_1(n, \nu) = \frac{1}{2}[n]_{(\epsilon q)^{-1}}(1 + (\epsilon q)^{1-n-2\nu}), \quad (4)$$

where $[n]_{(\epsilon q)^{-1}} = \frac{(\epsilon q)^{-n}-1}{(\epsilon q)^{-1}-1}$, and $n \in \mathbf{N}_0$, $\nu = 0, 1$.

It is important to observe that, for $q \in \mathbf{R}$, the function ϕ_1 is positive : $\phi_1(n, \nu) > 0$, $\forall n \in \mathbf{N}_0$, $\nu = 0, 1$. Furthermore, $\phi(n, \nu)$ cannot be written as a function of one variable. If it could be done so, this would mean that there would be only one tower of states, and that $a_1 \propto a_2$. Hence, all states $(a_1^\dagger)^n (a_2^\dagger)^\nu |0, 0\rangle$, eq.(3a), have positive definite norms and can be normalized. The normalized states are

$$|n, \nu\rangle = \frac{(a_1^\dagger)^n (a_2^\dagger)^\nu |0, 0\rangle}{\sqrt{[\phi_1(n, \nu)]!}}, \quad (5)$$

where $[\phi_1(n, \nu)]! = \phi_1(n, \nu) \cdots \phi_1(1, \nu)$, $[\phi_1(0, \nu)]! = 1$, and the orthonormality condition reads $\langle n, \nu | n', \nu' \rangle = \delta_{nn'} \delta_{\nu\nu'}$, $\nu, \nu' = 0, 1$. Owing to this orthonormality relation, any linear combination of states, eq.(5), has a positive norm. Specially,

$$||\alpha|n, 0\rangle + \beta|n-1, 1\rangle||^2 = |\alpha|^2 + |\beta|^2 > 0.$$

It is easy to find the action of the a_i, a_i^\dagger operators on the states ,eq.(5),namely:

$$a_1^\dagger |n, \nu\rangle = \sqrt{\phi_1(n+1, \nu)} |n+1, \nu\rangle, \quad (6)$$

$$a_1 |n, \nu\rangle = \sqrt{\phi_1(n, \nu)} |n-1, \nu\rangle,$$

$$a_2^\dagger |n, \nu\rangle = \sqrt{\frac{[\phi_1((n+2\nu), (1-\nu))]!}{[\phi_1(n, \nu)]!}} \left(\frac{1-\epsilon q}{\epsilon'(1+\epsilon q)}\right)^\nu (\epsilon'')^n |(n+2\nu), (1-\nu)\rangle,$$

$$a_2 |n, \nu\rangle = \sqrt{\frac{[\phi_1(n, \nu)]!}{[\phi_1((n-2+2\nu), (1-\nu))]!}} \left(\frac{1-\epsilon q}{\epsilon'(1+\epsilon q)}\right)^{1-\nu} (\epsilon'')^n |(n-2+2\nu), (1-\nu)\rangle.$$

In the picture (3a), the a_1^\dagger operator builds two infinite towers on $|0, 0\rangle$ and $|0, 1\rangle$, respectively, whereas the a_2, a_2^\dagger operators interconnect the two towers.

In the picture (3b), in which the indices are interchanged, $1 \leftrightarrow 2$ and $\varepsilon \leftrightarrow -\varepsilon$, the a_2^\dagger operator creates two towers based on $|0, 0\rangle$ and $|0, 1\rangle$ while the a_1^\dagger operator braids between these two towers. All equations (5-6) hold with $1 \leftrightarrow 2$, $\varepsilon \leftrightarrow -\varepsilon$.

Deformed SUSY oscillator. - We can define the operators $Q_{ij} = a_i a_j^\dagger$, ($Q_{ij}^\dagger = Q_{ji}$) and $\tilde{Q}_{ij} = a_i^\dagger a_j$, ($\tilde{Q}_{ij}^\dagger = \tilde{Q}_{ji}$), $i, j = 1, 2$, satisfying (in the picture (3a))

$$\begin{aligned}
Q_{ij} &= \delta_{ij} + p' R_{ki,jl} \tilde{Q}_{kl}, \\
[N, Q_{ij}] &= [N, \tilde{Q}_{ij}] = 0, \quad \forall i, j = 1, 2, \\
Q_{11}|n, \nu\rangle &= \phi_1(n+1, \nu)|n, \nu\rangle, \\
Q_{22}|n, \nu\rangle &= \phi_2(n+2\nu, 1-\nu)|n, \nu\rangle, \\
Q_{12}|n, \nu\rangle &= \psi_{12}(n, \nu)|n-1+2\nu, 1-\nu\rangle, \\
Q_{21}|n, \nu\rangle &= \psi_{21}(n, \nu)|n-1+2\nu, 1-\nu\rangle, \\
Q_{12}^\dagger Q_{12} &= Q_{21} Q_{12} = \psi_{12}^2(n, \nu), \\
Q_{12}^2 &= \psi_{12}(n, \nu) \psi_{12}(n-1+2\nu, 1-\nu),
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
\phi_2(n, \nu) &= \frac{[\phi_1(n, \nu)]!}{[\phi_1(n-2+2\nu, 1-\nu)]!} \left(\frac{1-\epsilon q}{\epsilon'(1+\epsilon q)} \right)^{2(1-\nu)}, \\
\psi_{12}(n, \nu) &= \sqrt{\frac{\phi_1(n-2+2\nu, 1-\nu)[\phi_1(n+2\nu, 1-\nu)]!}{[\phi_1(n, \nu)]!}} \left(\frac{1-\epsilon q}{\epsilon'(1+\epsilon q)} \right)^\nu (\epsilon'')^n \tag{8} \\
\psi_{21}(n, \nu) &= \psi_{12}(n-1+2\nu, 1-\nu).
\end{aligned}$$

Analogous relations can be obtained for the operators \tilde{Q}_{ij} and \tilde{Q}_{ij}^\dagger using eqs.(6,7).

Notice that $(Q_{12})^2 \neq 0$ ($(\tilde{Q}_{12})^2 \neq 0$) when $q^2 \neq 1$, and $(Q_{12})^2 = 0$ ($(\tilde{Q}_{12})^2 = 0$) when $q^2 = 1$.

We can define the Hamiltonian H as

$$\begin{aligned} \{Q_{12}, Q_{12}^\dagger\} &= 2H, \\ [H, Q_{12}] &= [H, Q_{12}^\dagger] = [H, N] = 0, \\ H|n, \nu\rangle &= \frac{1}{2}((\psi_{12}(n, \nu))^2 + (\psi_{12}(n-1+2\nu, 1-\nu))^2)|n, \nu\rangle, \end{aligned} \tag{9}$$

and similarly, the Hamiltonian \tilde{H} as

$$\begin{aligned} \{\tilde{Q}_{12}, \tilde{Q}_{12}^\dagger\} &= 2\tilde{H}, \\ [\tilde{H}, \tilde{Q}_{12}] &= [\tilde{H}, \tilde{Q}_{12}^\dagger] = [\tilde{H}, N] = 0, \\ \tilde{H}|n, \nu\rangle &= \frac{1}{2}((\tilde{\psi}_{12}(n, \nu))^2 + (\tilde{\psi}_{12}(n-1+2\nu, 1-\nu))^2)|n, \nu\rangle. \end{aligned} \tag{10}$$

The relations (9) and (10) define a new kind of q -deformation of the supersymmetric (SUSY) oscillator [9]. We point out that the spectrum of H (\tilde{H}) is positive and degenerate, i.e. the states $|n, 0\rangle$ and $|n-1, 1\rangle$ have the same energy $\frac{1}{2}((\psi(n, 0))^2 + (\psi(n-1, 1))^2)$. These properties are typical for SUSY oscillator, except that for $q^2 \neq 1$ the energy levels are not equidistant. In the limit $q = +1$, the state $|n, 0\rangle$ ($|n-1, 1\rangle$) is bosonic (fermionic) in the picture (3a). In the limit $q = -1$, the state $|0, n\rangle$ ($|1, n-1\rangle$) is bosonic (fermionic) in the picture (3b).

The q -deformed SUSY algebra (9) is generated by the set $\{N, Q_{12}, Q_{12}^\dagger, H\}$ and the q -deformed SUSY algebra (10) by the set $\{N, \tilde{Q}_{12}, \tilde{Q}_{12}^\dagger, \tilde{H}\}$. Notice that our Hamiltonian H (and \tilde{H}) is invariant under the q -superalgebra since H and Q (\tilde{H} and \tilde{Q}) mutually commute, in contrast to the Hamiltonian of the form $H = \{Q_+, Q_-\}$ mentioned in [10]. The q -deformed supercharges, operators $Q_{ij}, \tilde{Q}_{ij}, i \neq j$, also braid between the two towers and preserve the total number operator $N = N_1 + \nu$. Although

the operators Q and \tilde{Q} are not nilpotent ($Q_{12}^2 \neq 0$ for $q^2 \neq 1$, contrary to the ordinary SUSY oscillator), their irreducible representations remain two-dimensional, as a consequence of the relation $a_1^2 \propto a_2^2$ (eq.(1)).

Conclusion. - In conclusion, we have constructed and investigated the Fock-space representation for the "peculiar" algebra defined for a two-mode oscillator in ref.[8]. We have shown that this algebra corresponds to the deformed supersymmetric oscillator. This deformed SUSY oscillator represents an alternative mechanism for the violation of the Pauli exclusion principle [2]. It is also possible to generalize the "peculiar" algebra to a multimode case and to include arbitrary relations between powers of the operators a_i with arbitrary exponents.

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